A CLASS NUMBER RELATION IN FROBENIUS EXTENSIONS OF NUMBER FIELDS

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Let K/k be a normal extension of algebraic number fields whose Galois group G is a Frobenius group. Then K/k is said to be a Frobenius extension. Most of the structure of the unit group and of the ideal class group of K is determined by that of the subfields fixed by the Frobenius kernel N and by a complement F. Here this is investigated when G is a maximal or metacyclic Frobenius group. In particular, the results apply firstly to the normal closure of $k(\sqrt[p]{a})/k$ where $a \in k$ and p is a rational prime, and, secondly, when G is a dihedral group of order $2n$ for an odd integer n. A. Scholz, taking $n = p = 3$, was the first to consider this problem.

The first section describes some basic properties of the group ring $\mathbb{Z}[G]$ and the second section, which could be omitted in a preliminary reading, just serves to calculate a certain index in $\mathbb{Z}[G]$. The result is Theorem 2.1. In §3 the aim is to study the unit index Q which appears in the class number relation and a bound is obtained for it in Theorem 3.6. Then, in Theorem 4.4, the class number relation itself is derived. All the extraneous factors therein divide a power of the order n of N. This is explained in Theorem 5.3 by an underlying isomorphism between the maximal subgroups of the ideal class groups whose orders are prime to n.

The overall plan used to discover the class number relation is to eliminate the group of Minkowski units from R. Brauer's relation [**1**] and to calculate the consequent index in $\mathbb{Z}[G]$ by using regulators. When these ideas were first exhibited in an abstract of [**11**] at the Oberwolfach meeting in August 1975 discriminants were used instead of regulators, with the disadvantage that the index in $\mathbb{Z}[G]$ could be determined only for totally real fields. This restriction applies to W. Jehne's subsequent paper [**6**] on Frobenius extensions of $\mathbb{Q}$ with maximal type. The general case for maximal Frobenius groups had already occurred in [**9**], but reappears here together with the metacyclic case. Some more specific metacyclic extensions have been examined by F. Halter-Koch and N. Moser in [**2**,**3**,**4**, and **8**], while T. Honda in [**5**] has found the appropriate isomorphism of ideal class groups for general metacyclic Frobenius groups.

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§1. *Frobenius Groups*. Let G be a group with order $|G| = nf$ where n and f are co-prime and such that $g \in G$ implies $g^n = 1$ or $g^f = 1$. Suppose also that

$$N = \{g \in G \mid g^n = 1\}$$

is a proper normal subgroup of G. Then G is called a *Frobenius group* and N its *kernel*. Let $\tilde{S} \in \mathbb{Z}[G]$ denote the sum of the elements in a subset S of G. A *complement* of N is a subgroup F for which $\tilde{F}\tilde{N} = \tilde{G}$. There are precisely n such complements, which are conjugate under elements of N. They have order f and intersect pairwise in the identity, while N has order n. Hence

1.1 $$\tilde{N} + \sum \tilde{F} = \tilde{G} + n.\tilde{1},$$

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where the sum extends over all complements F. This implies

$$1_N^G + f.1_F^G \;=\; 1_1^G + f.1_G^G, \tag{1.2}$$

where 1_H^G denotes the character on G induced by the unit character on a subgroup H.

The centraliser of an element of $N - 1$ is contained in N. Hence $N - 1$ decomposes into orbits of length f under conjugation by elements of F and f divides n–1. Thus G is called *maximal* if $f = n - 1$. In this situation N is an abelian group of prime exponent. Now suppose G is metacyclic. Then both N and F are cyclic with generators ν and ϕ respectively, say, which satisfy a relation $\nu^r\phi = \phi\nu$. Here n must be odd. From this point, it is assumed that G is of one of these two types.

1.3 DEFINITION. *Let* $\{v_i \in N \mid 0 \le i \le f-1\}$ *be the set* $N - 1$ *when G is maximal and the set with* $v_i = v^i$ *when G is metacyclic. For the fixed complement* F_0, *generated by* ϕ *when G is metacyclic, let* $\sum'$ *and* $\prod'$ *denote sums and products over the f complements* $v_iF_0v_i^{-1}$.

Most other sums and products extend over the full set of n complements. Finally, for a left (*respectively* right) G-module X and a subgroup H of G let HX (*respectively* XH) be the subgroup of X fixed under the action of H. For example, NK and FK are the subfields of K fixed by N and F.

1.4 LEMMA. *Let Z be the intersection of* $\mathbb{Z}[N]$ *with the centre of* $\mathbb{Z}[G]$. *Then*

$$\mathbb{Z}[N] \;=\; \sum_i v_i Z$$

and this sum is direct up to elements in $\mathbb{Z}\tilde{N}$.

Proof. Z is generated by 1 and the elements $z_j = \sum_{h\in F} h^{-1}g_jh$ where the g_j are representatives of the $(n-1)/f$ conjugacy classes in $N - 1$. The equality comes from $1 + \sum_i v_i = \tilde{N} \in \bigcap_i v_iZ$ in the maximal case. For the metacyclic case, the minimum polynomial $\prod_{h\in F}(x - h^{-1}vh)$ of v over Z shows that v^f, and therefore any power of v, lies in $\sum_i v_iZ$. The directness is apparent from $\dim_{\mathbb{Z}} Z \;=\; 1 + (n-1)/f$.

1.5 THEOREM. *For any* $\mathbb{Z}[G]$-*module X define* $X' = \sum FX$. *Then X' is the* $\mathbb{Z}[G]$-*module generated by any FX and* $X' = \sum' FX$. *Also define* $X_0 = NX + X'$. *Then the sum* $X_0 = NX + \sum' FX$ *is direct up to elements whose nth multiple lies in GX. Moreover,* $nX \subset X_0$.

Proof. For $g \in N$ use 1.4 to choose $\alpha_i \in Z$ for which $g = \sum_i v_i\alpha_i$. If $x \in F_0X$ then $gx = \sum_i v_i\alpha_i x \in \sum' FX$. Thus $\sum' FX$ is a $\mathbb{Z}[G]$-module and contains every FX.

From 1.1 we have $nX \subset \tilde{N}X + \sum \tilde{F}X \subset X_0$. Also that equation yields

$$\mathbb{Q}[G] \;=\; N\mathbb{Q}[G] + \textstyle\sum' F\mathbb{Q}[G], \tag{1.6}$$

by the first part. A comparison of dimensions shows that this sum is direct up to elements in $\mathbb{Q}\tilde{G}$. Let $1 = e_N + \sum' e_F$ be a corresponding decomposition of 1 with $ne_N = \tilde{N}$ and $ne_F \in F\mathbb{Z}[G]$, say. Let $H, H' \in \{N, v_iF_0v_i^{-1}\}$ be distinct. Then $ne_H\tilde{H}' \in \mathbb{Z}\tilde{G}$ by decomposing $\tilde{H}'$ under 1.6. If $x_F \in FX$ for $F \ne H$ one finds that

$$ne_Hx_F \;=\; ne_H\left(\tilde{N} - \textstyle\sum_j \sum_{h\in F} hg_jh^{-1}\right)x_F \;=\; ne_H\tilde{N}x_F - \textstyle\sum_j ne_H\tilde{F}g_jx_F \;\in\; GX.$$

Similarly, when $x_N \in NX$ one obtains $ne_F x_N \in GX$ because $ne_F = \tilde{F}\alpha$ for some $\alpha \in \mathbb{Z}[N]$. Hence $ne_H x_{H'} \in GX$ if $x_{H'} \in H'X$. Consequently

$$nx_{H'} \; = \; \textstyle\sum_H ne_H\, x_{H'} \; \equiv \; ne_{H'}\, x_{H'} \text{ modulo } GX.$$

Suppose $\sum_H x_H = 0$ with $x_H \in HX$. Then

$$0 \; = \; ne_{H'} \textstyle\sum_H x_H \; \equiv \; ne_{H'}\, x_{H'} \; \equiv \; nx_{H'} \text{ modulo } GX$$

and $nx_{H'} \in GX$. Thus the sum for X_0 is direct as far as stated.

1.7 LEMMA. *Suppose G is metacyclic. Define $\beta_i \in \mathbb{Z}[G]$ by $(\nu-1)^i\beta_i = \tilde{F}_0(\nu-1)^i$. Then there is a direct sum decomposition of left $\mathbb{Z}[G]$-modules*

$$\mathbb{Z}[G]/N\mathbb{Z}[G] \; = \; \bigoplus_{0 \le i < f} \mathbb{Z}[N]\beta_i/N\mathbb{Z}[G].$$

Proof. Let β be the column vector $(\beta_0, \beta_1, \ldots, \beta_{f-1})^T$ and ϕ the column vector $(1, \phi, \phi^2, \ldots, \phi^{f-1})^T$. Then $M\phi = \beta$ for the matrix $M = (m_{ij})$ with $m_{ij} = (\nu^{r^j} - 1)^i / (\nu - 1)^i$. M is a Vandermonde matrix whose determinant is the unit $\prod_{i<j}(\nu^{r^j} - \nu^{r^i})/(\nu - 1)$ of $\mathbb{Z}[N]/\mathbb{Z}\tilde{N}$. Hence M is invertible and 1 may be expressed as a linear combination of the β_i's. The rest is now clear .

§2. *An Index Theorem.* Suppose C is a subgroup of order $c = 1$ or 2 generated by $\gamma \in G$. For any subgroup H and $g \in G$ write $HgC = \tilde{H}g\tilde{C}$ or $\frac{1}{2}\tilde{H}g\tilde{C}$ for the generators of $H\mathbb{Z}[G]C$ over $\mathbb{Z}$, and $|HgC| = |H||C|$ or $|H|$ respectively for their values under the unit character of G. Let $r_{2_\gamma}(H)$ be the number of such generators with $2|H|$ elements, and set $r_\gamma(H) = dim_{\mathbb{Z}}(H\mathbb{Z}[G]C/\mathbb{Z}\tilde{G})$.

2.1 THEOREM. *$\mathbb{Z}[G]C/(N\mathbb{Z}[G]C + \sum F\mathbb{Z}[G]C)$ has finite order $n^{fr_\gamma(N)/2}$ in the metacyclic case and $n^{(r_\gamma(N)+(f-1)(r_\gamma(F)-1))/2}$ in the maximal case. The exponent of the group is precisely n.*

The rest of the section is devoted to a proof of this. There are three possibilities for γ:

$$\gamma = 1, \qquad \gamma \in N - 1, \qquad \textit{or} \qquad \gamma \notin N.$$

Replacing γ by a conjugate does not change the order or the exponent of the quotient group. Thus if $\gamma \notin N$ it may be assumed that $\gamma \in F_0$. Because $F_0gC = \tilde{F}_0 g\tilde{C}$ for $g \in N - 1$ we have

$$2.2 \qquad \begin{array}{lclllll} r_{2_\gamma}(F) & = & 0, & n/2, & \textit{and} & (n-1)/2; & \\ r_\gamma(F) & = & n-1, & (n-2)/2, & \textit{and} & (n-1)/2; & \text{and} \\ r_\gamma(N) & = & f-1, & f-1, & \textit{and} & (f-2)/2, & \end{array}$$

respectively in three cases.

From the proof of 1.5, $n\tilde{C}$ decomposes in $N\mathbb{Z}[G]C + \sum F\mathbb{Z}[G]C$ with component $\tilde{N}\tilde{C}$ in $N\mathbb{Z}[G]C$. So the exponent is n for metacyclic groups. For G maximal 1.1 yields the explicit decomposition $n\tilde{C} = \tilde{N}\tilde{C} + \sum_i \tilde{F}_0(1-\nu_i)\tilde{C}$ and hence an exponent n.

The Metacyclic Case. For $\gamma = 1$ the required index is

$$\begin{aligned}\big[\mathbb{Z}[G]/N\mathbb{Z}[G] : {\textstyle\sum}'\mathbb{Z}[G]F/N\mathbb{Z}[G]\big] &= \big[\, {\textstyle\sum}_i \mathbb{Z}[G]\beta_i/N\mathbb{Z}[G] : {\textstyle\sum}_i \mathbb{Z}[G]F_0(\nu-1)^i/N\mathbb{Z}[G]\,\big] \\ &= {\textstyle\prod}_i \big[\, \mathbb{Z}[N]\beta_i/N\mathbb{Z}[G] : (\nu-l)^i\mathbb{Z}[N]\beta_i/N\mathbb{Z}[G]\,\big] \\ &= {\textstyle\prod}_i n^i \quad = \quad n^{f(f-1)/2}\end{aligned}$$

by 1.7. Otherwise the assumption $\gamma = \phi^{f/2}$ holds. Let $A_i = \tilde{C}\,\mathbb{Z}[G]\beta_i/N\mathbb{Z}[G]$. Then β_i may be replaced by

$$\beta_i' = \left(\frac{\nu}{\nu+1}\right)^i \beta_i$$

to give $(\nu^j + (-1)^i\nu^{-j})\beta_i'$ with $1 \le j \le (n-1)/2$ as a basis of A_i over $\mathbb{Z}$. $A_i \oplus \nu A_i$ is a $\mathbb{Z}[G]$-module because if $\alpha \in A_i$ then $\nu^2\alpha = -\alpha + \nu(\nu + \nu^{-1})\alpha \in A_i \oplus \nu A_i$. When i is even,

$$\beta_i' = -{\textstyle\sum}_j (\nu^j + \nu^{-j})\, \beta_i' \in A_i \oplus \nu A_i \qquad \text{so that} \qquad A_i \oplus \nu A_i = \mathbb{Z}[G]\beta_i/N\mathbb{Z}[G].$$

When i is odd,

$$(\nu - \nu^{-1})\beta_i' \in A_i \oplus \nu A_i \qquad \text{so that} \qquad A_i \oplus \nu A_i = (\nu - 1)\mathbb{Z}[G]\beta_i/N\mathbb{Z}[G],$$

and this has index n in $\mathbb{Z}[G]\beta_i/N\mathbb{Z}[G]$. Hence if

$$B = {\textstyle\sum}_i A_i = C\mathbb{Z}[G]/N\mathbb{Z}[G]$$

then $B \oplus \nu B$ has index $n^{f/2}$ in $\mathbb{Z}[G]/N\mathbb{Z}[G]$. $A_0 \oplus \nu A_0 = \mathbb{Z}[G]F_0/N\mathbb{Z}[G]$ shows that if $D = \sum C\mathbb{Z}[G]F/N\mathbb{Z}[G]$ then $D \oplus \nu D = \sum \mathbb{Z}[G]F/N\mathbb{Z}[G]$. Thus the required index $q = [B : D]$ is given by

$$\begin{aligned}n^{f(f-1)/2} &= [Z[G]/N\mathbb{Z}[G] : {\textstyle\sum} \mathbb{Z}[G]F/N\mathbb{Z}[G]] \\ &= n^{f/2}\,[B \oplus \nu B : D \oplus \nu D] \\ &= n^{f/2}q^2 .\end{aligned}$$

The Maximal Case. When G is maximal the techniques of [**11**] are suitable for the order calculation. Define a pairing on $\mathbb{Z}[G] \times \mathbb{Z}[G]$ by $(x, y) = |G|^{-1}1_1^G(xy^*)$ where $*$ is the involution induced by $g \mapsto g^{-1}$ for $g \in G$. If X is a subgroup of $\mathbb{Z}[G]$ with basis $\{x_i\}$ let

$$R(X) = \big|\det\big((x_i, x_j)\big)\big|$$

be the regulator of X. This is independent of the choice of basis.

2.3 LEMMA. *If* $X = \sum F\mathbb{Z}[G]C$ *then* $R(X) = fn^{(f-1)(r_\gamma(F)-1)}2^{fr_{2\gamma}(F)}$.

Proof. Let $g, g' \in N - 1$ be fixed. Then $ghg'h' = 1$ implies $h' = h^{-1}$ for $h, h' \in F$. But $ghg'h^{-1} = 1$ has only one solution $h \in F$. Hence $g\tilde{F}g'\tilde{F}$ contains the identity once. If $g \in N - 1$ and $g' = 1$, or $g' \in N - 1$ and $g = 1$, then 1 does not appear in $g\tilde{F}g'\tilde{F}$, but it occurs f times for $g = g' = 1$.

Choose $S \subseteq N - 1$ so that $\{FsC \mid s \in S \text{ or } s = 1\}$ is a basis of $F\mathbb{Z}[G]C$. If $t, t' \in N - 1$ and $s, s' \in S$ then $(t'Fs'C, tFsC)$ is c times the multiplicity of l in $t^{-1}t'\tilde{F}s'\tilde{C}s^{-1}\tilde{F}$. Since $s'\gamma s^{-1} \notin F$ for $c = 2$ the value of the pairing is given by:

	$t = t'$	$t \neq t'$
$s = s'$	cf	$c^2 - c$
$s \neq s'$	0	c^2

Also $(\tilde{G}, \tilde{G}) = nf$ and $(\tilde{G}, t\tilde{F}s\tilde{C}) = cf$. Take $\{\tilde{G}, tFsC \mid t \in N-1, s \in S\}$ for a basis of X. The corresponding matrix for $R(X)$ includes $|S| \times |S|$ blocks, one for each pair $t, t' \in N - 1$. Observe that $|S| = r_\gamma(F)$ and let J, J_r, and J_c be the $r_\gamma(F) \times r_\gamma(F)$, $1 \times r_\gamma(F)$, and $r_\gamma(F) \times 1$ matrices consisting entirely of unit entries. Then the regulator may be calculated as follows :

$$R(X) = \begin{vmatrix} nf & cfJ_r & cfJ_r & \cdots & cfJ_r \\ cfJ_c & cfI & c^2J - cI & \cdots & c^2J - cI \\ cfJ_c & c^2J - cI & cfI & \cdots & c^2J - cI \\ \vdots & \vdots & \vdots & & \vdots \\ cfJ_c & c^2J - cI & c^2J - cI & \cdots & cfI \end{vmatrix}$$

$$= \begin{vmatrix} nf & 0 & 0 & \cdots & cfJ_r \\ cfJ_c & cnI - c^2J & 0 & \cdots & c^2J - cI \\ cfJ_c & 0 & cnI - c^2J & \cdots & c^2J - cI \\ \vdots & \vdots & \vdots & & \vdots \\ cfJ_c & c^2J - cnI & c^2J - cnI & \cdots & cfI \end{vmatrix}$$

$$= |cnI - c^2J|^{n-2} \begin{vmatrix} nf & cfJ_r \\ cf^2J_c & cI + c^2(n-2)J \end{vmatrix}$$

$$= \{(cn)^{r_\gamma(F)-1}(cn - c^2r_\gamma(F))\}^{f-1}f \begin{vmatrix} n & cJ_r \\ cJ_c & cI \end{vmatrix}$$

$$= fn^{(f-1)(r_\gamma(F)-1)}(n - cr_\gamma(F))^f c^{fr_\gamma(F)}$$

$$= fn^{(f-1)(r_\gamma(F)-1)}2^{fr_{2\gamma}(F)}, \qquad \text{by 2.2.}$$

Let $\rho : \mathbb{Z}[G]C \to L_\gamma = \mathbb{Z}[G]C/\mathbb{Z}\tilde{G}$ be the natural map and define a pairing on $L_\gamma \times L_\gamma$ by $(\rho x, \rho y) = |G|^{-1}(1_1^G - 1)(xy^*)$ for $x, y \in \mathbb{Z}[G]C$ and the involution $* : g \in G \mapsto g^{-1}$. Suppose X is a subgroup of L_γ. Take $\{x_j\}$ in $\mathbb{Z}[G]C$ such that $\{\rho x_j\}$ is a basis of X. Then $\{\tilde{G}, x_j\}$ is a basis of $\rho^{-1}X$. So

$$R(\rho^{-1}X) = \left\| \begin{matrix} (x_i, x_j) & (x_i, \tilde{G}) \\ (\tilde{G}, x_j) & (\tilde{G}, \tilde{G}) \end{matrix} \right\|_{i,j} .$$

But $(\rho x, \rho y) = (x, y) - |G|^{-1}(x, \tilde{G})(\tilde{G}, y)$ for $x, y \in \mathbb{Z}[G]C$. Hence row operations give $R(\rho^{-1}X) = |G|R(X)$ for the obvious definition of $R(X)$. Now 2.3 yields

2.4 LEMMA. $R(\sum FL_\gamma) = n^{(f-1)r_\gamma(F)-f}2^{fr_{2\gamma}(F)}$.

If $x, y \in \mathbb{Z}[G]C$ satisfy $\rho x \in NL_\gamma$ and $\rho y \in \sum FL_\gamma$ then $(\rho x, \rho y) = 0$. Also the sum $NL_\gamma + \sum' FL_\gamma$ is direct by 1.5. Thus,

2.5 LEMMA. $R(NL_\gamma + \sum FL_\gamma) = R(NL_\gamma)\, R(\sum FL_\gamma)$.

From [**11**], 3.5 and 3.3, the following facts may be recalled :

2.6 $\quad R(HL_\gamma) = |G|^{-1}|H|^{r_\gamma(H)+1}2^{r_{2\gamma}(H)}$ *for a subgroup* H;

2.7 $\quad R(Y) = [X : Y]^2 R(X)$ *for subgroups* X, Y *of* L_γ *for which* $[X : Y]$ *is defined.*

Combining equations 2.4–2.7 for $X = L_\gamma$ and $Y = NL_\gamma + \sum FL_\gamma$ gives the order of

$$\mathbb{Z}[G]C \,/\, (N\mathbb{Z}[G]C + \textstyle\sum F\mathbb{Z}[G]C)$$

as

$$[X : Y] = \{R(NL_\gamma)\, R(\textstyle\sum FL_\gamma)/R(L_\gamma)\}^{\frac{1}{2}} = \{n^{r_\gamma(N)+(f-1)(r_\gamma(F)-1)}\}^{\frac{1}{2}}.$$

The power of 2 is eliminated by observing that $r_{2_\gamma}(H) = 1_H^G\,(1-\gamma)/2$ and evaluating 1.2 at $(1-\gamma)/2$.

§3. *The Unit Group*. Suppose the normal extension K/k of number fields has the Frobenius group G as its Galois group. Let U and W be the groups of units and roots of unity in K.

3.1 LEMMA. $W = NW$ *and* $FW = GW$.

Proof. Let $H = \mathrm{Gal}(K/k(W))$. Then H is normal in G and G/H is abelian. The former property implies $H \subset N$ or $N \subseteq H$. However, if $H \subset N$ then G/H is Frobenius and therefore not abelian. Thus $N \subseteq H$ and $W = NW$. Now set $H' = \mathrm{Gal}(K/k(FW))$. Then, similarly, $N \subseteq H'$. But $F \subseteq H'$ also. Therefore $H' = G$ and $FW = GW$.

The unit group U will be written additively when the notation makes this more convenient. In particular, for a subgroup H of G let $\mathbb{Q}HU$ be the subgroup of units with some non-trivial multiple (*i.e.* power) fixed by H. Define

3.2 $$I(H) = [HU \cap \mathbb{Q}GU : GU + HW].$$

It was shown in [**11**, §4] that $I(H)$ divides $[G : H]$.

3.3 LEMMA. $\mathbb{Q}GU = N\mathbb{Q}GU + F\mathbb{Q}GU$ *and the sum is direct up to elements in* GU. *Hence* $I(1) = I(N)I(F)$.

Proof. The three groups modulo $GU + W$ have orders $I(1)$, $I(N)$, and $I(F)$ which divide nf, f, and n respectively. The sum is therefore direct because $(n, f) = 1$. Choose $a, b \in \mathbb{Z}$ such that $an + bf \equiv 1 \bmod nf$. Take $\varepsilon \in \mathbb{Q}GU$ and write $[\varepsilon]$ for its class modulo $GU + W$. Since G acts trivially on $\mathbb{Q}GU/(GU + W)$ it follows that $[\varepsilon] = [(an + bf)\varepsilon] = [\tilde{N} a\varepsilon] + [\tilde{F} b\varepsilon] \in (N\mathbb{Q}GU + F\mathbb{Q}GU)/(GU + W)$.

Application of 1.5 shows that $Q = [U : U_0]$ is finite and divides a power of n. Theorem 4.1 of [**11**] proves that $[\mathbb{Q}FU : FU + W]$ divides f, which is prime to n, and $FU + W \subset U_0$. Therefore

3.4 $$\mathbb{Q}FU \subset U_0\,.$$

Now the directness of 1.5 for U together with 3.3 yield

3.5 LEMMA. $\mathbb{Q}FU = FU + N\mathbb{Q}GU$.

3.6 THEOREM. *Let $r(H)$ be the rank of HU/HW. Then $Q = [U : U_0]$ divides $In^{(f-1)(r(F)-r(G))}$ for $I = [\mathbb{Q}NU : \mathbb{Q}NU \cap U_0]$ and I in turn divides n.*

Proof. For any $\mathbb{Z}[G]$-module X the quotient $X/\mathbb{Q}HX$ is torsion-free. Take $x \in X$ with image in $H(X/\mathbb{Q}HX)$. Then $(|H| - \tilde{H})x \in \mathbb{Q}HX$ and so $x \in \mathbb{Q}HX$. Thus $\mathbb{Q}H(X/\mathbb{Q}HX) = 0$. In particular, $V = U/\mathbb{Q}NU$ has $\mathbb{Q}GV \subset \mathbb{Q}NV = 0$. Hence $V_0 = \sum' FV$ and 1.5 shows this sum is direct. If $\varepsilon \in U$ has image in $\mathbb{Q}FV$ then $\tilde{F}\varepsilon - f\varepsilon \in \mathbb{Q}NU$. So $[FV : (FU + \mathbb{Q}NU)/\mathbb{Q}NU]$ divides a power of f and the same is true of $[V_0 : (U_0 + \mathbb{Q}NU)/\mathbb{Q}NU]$. However, the latter index divides Q and thus a power of n. Therefore $V_0 = (U_0 + \mathbb{Q}NU)/\mathbb{Q}NU$ and $Q = [U : U_0] = [V : V_0]I$. From 1.5 the exponent of V/V_0 divides n. Also $\mathbb{Q}FV = FV$ and the rank of V_0/FV is $(f-1)(r(F) - r(G))$. Thus the index $[V : V_0] = [V/FV : V_0/FV]$ divides $n^{(f-1)(r(F)-r(G))}$. Finally I divides n by Theorem 4.1 of [**11**].

3.7 LEMMA. *The norms $N_{K/FK}\,U = \tilde{F}U$ satisfy $FU = \tilde{F}U + GU$.*

Proof. Let S be a set of representatives for the conjugacy classes of $N - 1$ under F. If $\varepsilon \in FU$ then

$$\varepsilon = \left(\tilde{N} - \sum_{h\in F}\sum_{g\in S} hgh^{-1}\right)\varepsilon = \tilde{N}\varepsilon - \tilde{F}\tilde{S}\varepsilon \in GU + \tilde{F}U\,.$$

§4. *The Class Number Relation.* Let $\{C_i\}$ be the set of decomposition groups in G for one prime divisor in K of each of the $r = r(G) + 1$ infinite primes in k. They are defined up to conjugacy which depends on the chosen embedding of K into **C**. Suppose L and L_i satisfy the exact sequences of $\mathbb{Z}[G]$-modules

$$0 \to \mathbb{Z} \to \bigoplus_{i=1}^{r} \mathbb{Z}[G]C_i \to L \to 0,$$

where $n \in \mathbb{Z} \mapsto n \oplus_i \tilde{G}$; and

$$0 \to \mathbb{Z} \to \mathbb{Z}[G]C_i \to L_i \to 0,$$

where $n \in \mathbb{Z} \mapsto n\tilde{G}$. In both cases let G act trivially on $\mathbb{Z}$. Both sequences are exact when fixed under the action of a subgroup H. Let $L_0 = NL + \sum FL$; $L_{i0} = NL_i + \sum FL_i$; $Q^* = [L : L_0]$; and $Q_i^* = [L_i : L_{i0}]$. Then Q^* and Q_i^* are finite by Theorem 1.5. Moreover ,

$$\begin{aligned} L/L_0 &\cong \{\oplus_i \mathbb{Z}[G]C_i\}/\{\oplus_i (N\mathbb{Z}[G]C_i + \textstyle\sum F\mathbb{Z}[G]C_i)\} \\ &\cong \oplus_i\{(\mathbb{Z}[G]C_i)/(N\mathbb{Z}[G]C_i + \textstyle\sum F\mathbb{Z}[G]C_i)\} \cong \oplus_i L_i/L_{i0}. \end{aligned}$$

Consequently,

4.1 $$Q^* = \prod_{i=1}^{r} Q_i^*\,.$$

The index $Q_i{*}$ is just the order of the group in Theorem 2.1 with $C = C_i$. If $r_i(H) = \dim HL_i$ then $\sum_i (r_i(H)+1) = \dim HL + 1 = r(H) + 1$ is the number of infinite primes in HK. Thus $r(H)$ is the rank of the unit group HU/HW. Combining 2.1 with 4.1 yields:

4.2 LEMMA. $Q^* = n^{f(r(N)-r(G))/2}$ *in the metacyclic case and*

$Q^* = n^{(r(N)-r(G)+(f-1)(r(F)-2r(G)-1))/2}$ *in the maximal case.*

Let a bar denote the canonical map $U \to U/W$ and choose a submodule M of $\overline{U}$ which is $\mathbb{Z}[G]$-isomorphic to L. Recall the definitions of $I(H)$ and Q in 3.2 and 3.6.

4.3 LEMMA.

$$\frac{[\overline{U}:M][\overline{GU}:GM]^f}{[\overline{NU}:NM][\overline{FU}:FM]^f} = \frac{Q}{Q^*I(F)^{f-1}}.$$

Proof. Begin by observing that $(\mathbb{Q}GU \cap HU)/W = G\overline{HU}$ so that $I(H) = [G\overline{HU} : \overline{GU}]$. Also $G\overline{U} = \mathbb{Q}GU/W = (\mathbb{Q}GU \cap U_0)/W = \overline{GU_0}$ by 3.3. For convenience, let $V = G\overline{U}$. Then

$$
\begin{aligned}
&Q^*\,[\overline{U} : M] \,/\, [\overline{GU} : GM]\, I(1)\, Q \\
&\quad= [\overline{U}_0 : M_0][GM : V] = [\overline{U}_0/V : (M_0 + V)/V] \\
&\quad= [(\overline{NU} + V)/V : (NM + V)/V] \textstyle\prod{}'[(\overline{FU} + V)/V : (FM + V)/V] \\
&\quad= [\overline{NU} : NM + G\overline{NU}][\overline{FU} : FM + G\overline{FU}]^f \\
&\quad= [\overline{NU} : NM][\overline{FU} : FM]^f / [G\overline{NU} : NM \cap G\overline{NU}][G\overline{FU} : FM \cap G\overline{FU}]^f \\
&\quad= [\overline{NU} : NM][\overline{FU} : FM]^f / I(N)I(F)^f [\overline{GU} : GM]^{f+1}.
\end{aligned}
$$

Now apply 3.3.

4.4 THEOREM. *Suppose the normal extension K/k of number fields has a maximal or metacyclic Frobenius group G as its Galois group. Let $h(H)$ be the class number and $r(H)$ the rank of the unit group HU of the subfield fixed by a subgroup H of G. If the kernel N and a complement F have orders n and f respectively then*

$$\frac{h(1)h(G)^f}{h(N)h(F)^f} = QI(F)^{1-f}n^{-A},$$

where

$Q = [U : NU \prod FU]$ *with* $\prod$ *over the complements* F;

$I(F)$, *defined in* 3.2, *is the order of* $(FU / GU)_{\mathrm{tor}}$ *and divides* n;

$A = \frac{1}{2}\{r(N) - r(G) + (f-1)(r(F) - 2r(G) + 1)\}$ *in the maximal case*; *and*

$A = (f-1) + \frac{1}{2} f\,(r(N) - r(G))$ *in the metacyclic case.*

The quotient group U/U_0 defining Q has exponent dividing n and it has order bounded by 3.6. *The product $U_0 = NU \prod{}' FU$ defined in* 1.3 *is direct up to units whose nth powers lie in k.*

Proof. The form of Brauer's class number relation [**1**] which is required here is given in [**10**, Theorem 4.1]. This shows that

$$\frac{h(1)h(G)^f}{h(N)h(F)^f} = \frac{nf\,|W|[\overline{U}:M]|GW|^f\,[\overline{GU}:GM]^f}{f\,|NW|[\overline{NU}:NM]n^f[\overline{FU}:FM]^f} = \frac{n^{1-f}Q}{Q*I(F)^{f-1}}$$

by 3.1 and 4.3. Now 4.2 yields the stated relation.

§5. *The Class Groups.* Let $C(H)$ be the part of the ideal class group of HK formed from the classes whose orders are prime to n.

5.1 THEOREM. *For any Frobenius group the following sequence is exact under the maps induced by extension of ideals.*

$$0 \to C(G) \to C(F) \to FC(1)/GC(1) \to 0.$$

Proof. The sequence is exact at $C(G)$ because $C(G)$ has order prime to the degree n of FK/GK. The two central maps compose to give the zero map. Suppose $\mathscr{C}$ is a class of $C(F)$ which maps into $GC(1)$. It is necessary to show that if $\mathfrak{a}$ is an ideal such that $\mathfrak{a}^n \in \mathscr{C}$ then the class of the norm $N_{FK/GK}\,\mathfrak{a}$ in $C(G)$ maps to $\mathscr{C}$. This will establish the exactness at $C(F)$. Let us consider all ideals to be extended to K and write the group of such ideals additively. Then $(g-1)\mathfrak{a}$ is principal for $g \in G$ because the image of $\mathscr{C}$ in $C(1)$ is fixed by G. Suppose $(g-1)\mathfrak{a} = (\alpha_g)$. If $h \in F$ then

$$(\alpha_g) = (g-1)\mathfrak{a} = (g-1)h\mathfrak{a} = h(h^{-1}gh-1)\mathfrak{a} = h(\alpha_{h^{-1}gh}).$$

Thus it may be assumed that $h\alpha_{h^{-1}gh} = \alpha_g$ and $\alpha_1 = 1$. Let S be a set of representatives for the conjugacy classes of $N-1$ under F. Then

$$(\tilde{N}-n)\,\mathfrak{a} = \sum_{g\in N}(\alpha_g) = \left(\sum_{g\in S}\sum_{h\in F}h^{-1}\alpha_g\right) = \left(\sum_{g\in S}\tilde{F}\alpha_g\right)$$

which is the extension of a principal ideal of FK. Finally, to prove the surjectivity, let $\mathscr{C}'$ be a class of $FC(1)/GC(1)$ and $\mathfrak{a}$ an ideal whose image is in $\mathscr{C}'$. With S as above,

$$\mathfrak{a} = \left(\tilde{N} - \sum_{h\in F}\sum_{g\in S}hgh^{-1}\right)\mathfrak{a} \sim (\tilde{N} - \tilde{F}\tilde{S})\,\mathfrak{a}$$

where ~ is equality up to a principal ideal. Thus the ideal $-\tilde{F}\tilde{S}\,\mathfrak{a}$ in $C(F)$ has image in $\mathscr{C}'$ because $\tilde{N}\,\mathfrak{a}$ is in a class of $GC(1)$. Hence the map is surjective and this completes the proof. Theorem 1.5 yields :

5.2 LEMMA. *Let X be a $\mathbb{Z}[G]$-module such that the order of X/GX is finite and prime to n. Then there is a direct sum decomposition*

$$X/GX = NX/GX + \sum{}' FX/GX.$$

5.3 THEOREM. *The maximal subgroups $C(H)$ of the ideal class groups of the HK with orders prime to n satisfy*

$$C(1)/C(N) \cong \bigoplus_{i=1}^{f} C(F)^{(i)}/C(G)\,,$$

where $C(F)^{(i)} \cong C(F)$ *and the embeddings* $C(N) \hookrightarrow C(\mathrm{l})$ *and* $C(G) \hookrightarrow C(F)^{(i)}$ *are induced by extension of ideals.*